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Complexity of Blowup Problems – Extended Abstract –

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Abstract

Consider the initial value problem of the first-order ordinary differential equation

$$\frac{d}{dt}x(t) = f(t, x(t)), \quad x(t_0) = x_0$$

where the locally Lipschitz continuous function $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$ with open domain and the initial datum $(t_0, x_0) \in \mathbb{R}^{l+1}$ are given. It is shown that the solution operator producing the maximal "time" interval of existence and the solution on it is computable. Furthermore, the complexity of the blowup problem is studied for functions f defined on the whole space. For each such function f the set Z of initial conditions (t_0, x_0) for which the positive solution does not blow up in finite time is a G_δ-set. There is even a computable operator determining Z from f. For $l \geq 2$ this upper G_δ-complexity bound is sharp. For $l = 1$ the blowup problem is simpler.

Keywords: Type-2 theory, differential equation, blowup.

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1 Indroduction

Consider an initial value problem of obtaining solutions $x(t)$ to the first-order ODE (*ordinary differential equation*)

$$\begin{cases} \frac{d}{dt}x(t) = f(t, x(t)),\ t \in \mathbb{R},\ (t, x) \in E \subseteq \mathbb{R}^{l+1} \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where the initial datum $(t_0, x_0) \in E \subseteq \mathbb{R}^{l+1}$ and the (generally nonlinear) function $f : E \to \mathbb{R}^l$ are given. In this initial-value problem, x is usually referred to as the space variable and t the time variable. If f is continuous on E and locally Lipschitz continuous in space variable x, then the problem (1) has a unique solution on a maximal time interval (α, β). This result is commonly referred to as Picard-Lindelöf existence and uniqueness theorem. Various versions of the computable Picard-Lindelöf theorem have been studied by several authors, including Aberth [1,2], Bishop and Bridges [5], Graça, Zhong and Buescu [11], Ko [15], Pour-El and Richards [19]. In this paper, we present a fully uniform version of the Picard-Lindelöf theorem.

The Picard-Lindelöf theorem gives a very satisfactory local theory for the existence and uniqueness of solutions to the ODE (1) for locally Lipschitz continuous f. However, there remains a difficult issue: Whether the corresponding maximal interval of existence (α, β) is bounded or not for any given initial datum. When β or/and α is finite, the solution $x(t)$ will blow up in finite time in the sense that $||x(t)||$ approaches to infinity as $t \to \beta^-$ or $t \to \alpha^+$. In general, it is difficult to predict whether or not a solution will blow up for a given initial datum, because it often requires extra knowledge on some quantitative estimates and asymptotics of the solution over long period of time, such as whether the solution satisfies a certain "coercive" conservation law. Indeed, it is shown recently in [11] and [7] that the blowup problem cannot be solved by any algorithm.

In this paper, we study the complexity of the blowup problem for functions f defined on the whole space. We shall use the notation CBU_f to denote the set of all initial data at which the solutions to the initial-value problem (1) are global (no blowup). The complement of CBU_f, denoted as BU_f, is then the set of all initial data for which the solutions blow up. We show that the set CBU_f is a G_δ set, i.e a countable intersection of open sets, and there is an algorithm that computes CBU_f from f. Thus the blowup set BU_f has F_σ as an upper complexity bound. Moreover, for every computable G_δ-set G of $\mathbb{R}^{l-1}$ with $l \geq 2$, we show that there exists a computable and effectively locally Lipschitz function $f : \mathbb{R}^l \to \mathbb{R}^l$ such that the solution to the problem "$x'(t) = f(x(t), x(0) = (x_0, 0)$" is global if and only if $x_0 \in G$. In other words, the G_δ-complexity for CBU_f is sharp. It follows that the F_σ-complexity is sharp for the blowup sets.

The paper is organized as follows. Section 2 introduces necessary concepts and results from computable analysis. Section 3 presents a fully uniform version of the computable Picard-Lindelöf theorem. Section 4 contains the theorems on the complexity of the blowup problem. We omit most of the proofs of Section 3. Detailed proofs can be found in the full version of this paper.

2 Preliminaries

For studying computability in analysis, in this article we use the representation approach also called type-2 theory of effectivity (TTE) [23]. In this theory computability on finite or infinite sequences, Σ^* or Σ^ω, respectively, over a finite alphabet Σ is defined explicitly by type-2 machines, which are Turing machines with finite or infinite one-way input and output tapes. The elements of Σ^* or Σ^ω are used as "names" of natural, rational or real numbers, of open sets, continuous functions and so on. A *representation* of a set M is a surjective partial function $\delta :\subseteq Y \to M$ ($Y \in \{\Sigma^*, \Sigma^\omega\}$), where p is called a δ-name or a name of $x \in M$ if $\delta(p) = x$. (In [23] representations $\delta :\subseteq \Sigma^* \to M$ are called *notations*.) A function on represented spaces is computable, if it can be realized by a computable function on the names.

We also use the more general *multi-representations* $\delta : Y \rightrightarrows M$, where $p \in Y$ is considered as a name of each $x \in \delta(p)$ and multi-functions $f : M \rightrightarrows M'$ on represented sets, where $y \in f(x)$ can be interpreted as "y is an acceptable result on input x". For multi-representations $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$, a function $h :\subseteq Y \to Y'$ realizes a multi-function $f : M \rightrightarrows M'$, if $h(p)$ is a γ'-name of some $y \in f(x)$ whenever $p \in \Sigma^\omega$ is a γ-name of x (see Figure 1). We call the multi-function f (γ, γ')-computable (-continuous), if it has a computable (continuous) realization [24,25].

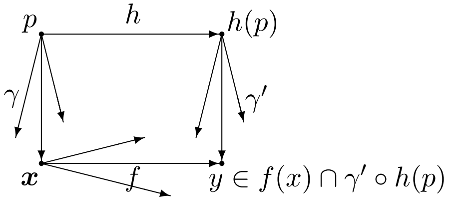

Fig. 1. $h(p)$ is a name of some $y \in f(x)$, if p is a name of $x \in \mathrm{dom}(f)$.

The extension to multi-functions $f : M_1 \times \ldots \times M_k \rightrightarrows M'$ is straightforward. For multi-representations $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$, $\gamma \le \gamma'$ (γ is reducible to γ'), if here is a computable function $h :\subseteq Y \to Y'$ such that $\gamma(p) \subseteq \gamma' \circ h(p)$ for all $p \in \mathrm{dom}(\gamma)$. The representations are equivalent, if they are reducible to each other. Equivalent representations induce the same computability and relative continuity on the represented set.

If multi-functions on represented sets have realizations, then their composition is realized by the composition of the realizations. Therefore, the computable multi-functions on represented sets are closed under composition. Much more generally, the computable multi-functions on represented sets are closed under flowchart programming with indirect addressing [24,25]. We will apply this result repeatedly, which allows convenient informal constructions of new computable functions on multi-represented sets from given ones.

Let $\gamma : Y \rightrightarrows M$ and $\gamma' : Y' \rightrightarrows M'$ be multi-representations. By means of computable standard pairing and tupling functions on Σ^* and Σ^ω, all of which we denote by $\langle\ \rangle$ [23], multi-representations of products can be defined: $[\gamma,\gamma']\langle y,y'\rangle := \gamma(y)\times\gamma'(y')$ and $\gamma^\omega\langle y_0,y_1,\ldots\rangle := \gamma(y_0)\times\gamma(y_1)\times\ldots$.

In [25] a multi-representation $[\gamma \rightrightarrows \gamma']$ of the (γ,γ')-continuous multi-functions $f : M \rightrightarrows M'$ is defined by $f \in [\gamma \rightrightarrows \gamma'](p)$, if η_p realizes f ($\eta_p = h$ in Figure 1). Here η is the canonical representation of the continuous functions $h :\subseteq Y \to Y'$ with open domain (for $Y' = \Sigma^*$) or G_δ-domain (for $Y' = \Sigma^\omega$) [23]. Its restrictions to the partial functions and total functions are called $[\gamma \to_p \gamma']$ and $[\gamma \to \gamma']$, respectively.

Let $\gamma_0 :\subseteq Y_0 \rightrightarrows M_0$ be another multi-representation. For a multi-function $f : M_0 \times M \rightrightarrows M'$ define $Tf(x)(y) := f(x,y)$. Then T is $([[\gamma_0,\gamma] \rightrightarrows \gamma'],[\gamma_0 \to [\gamma \rightrightarrows \gamma'])$-computable and its inverse is $([\gamma_0 \to [\gamma \rightrightarrows \gamma'],[[\gamma_0,\gamma] \rightrightarrows \gamma'])$-computable. As corollaries,

$$f \text{ is } (\gamma_0,\gamma,\gamma')\text{-computable} \iff Tf \text{ is } (\gamma_0,[\gamma \rightrightarrows \gamma'])\text{-computable}, \tag{2}$$

and for every multi-representation δ of multi-functions $h : M \rightrightarrows M'$, the evaluation $(h,x) \mapsto h(x)$ is (δ,γ,γ')-computable, iff $\delta \le [\gamma \rightrightarrows \gamma']$ [25] (cf. the special case for single-valued representations and total functions [23, Theorem 3.3.15]).

Let $\nu_{\mathbb{N}}$ and $\nu_{\mathbb{Q}}$ be standard notations of the natural numbers and the rational numbers, respectively. For single-valued representations $\gamma :\subseteq Y \to M$, $\gamma^\omega \equiv [\nu_{\mathbb{N}} \to \gamma]$ (representation of sequences on M).

On the space $\mathbb{R}^n$ we use the maximum norm

$$\|(x_1,\ldots,x_n)\| := \max\{|x_1|,\ldots,|x_n|\}.$$

For $x \in \mathbb{R}^n$ and $r > 0$ let $B(x,r) := \{y \in \mathbb{R}^n \mid \|x-y\| < r\}$ be the open ball or cube with center x and radius r. Let I^n be a natural notation of the set of all rational open balls $\mathrm{RB}^n := \{B(x,r) \mid x \in \mathbb{Q}^n,\ \ r \in \mathbb{Q},\ \ r > 0\}$ in $\mathbb{R}^n$. Let $\rho^n :\subseteq \Sigma^\omega \to \mathbb{R}^n$ be the representation defined by $\rho^n(p) = x$, iff p is a list of *all* open balls $J \in \mathrm{RB}^n$ (encoded by I^n) such that $x \in J$. Then $\rho := \rho^1$ is equivalent to the Cauchy representation of the real numbers [23]. For the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,\infty\}$, the "lower representation" $\overline{\rho}_< : \Sigma^\omega \to \mathbb{R}$ and the "upper representation" $\overline{\rho}_> : \Sigma^\omega \to \mathbb{R}$ are defined by $\overline{\rho}_<(p) = \sup\{r \in \mathbb{Q} \mid r \text{ is listed by } q\}$. and $\overline{\rho}_<(p) = \inf\{r \in \mathbb{Q} \mid r \text{ is listed by } q\}$.

For the set $O(\mathbb{R}^n)$ of open subsets and the set $G_\delta(\mathbb{R}^n)$ of the G_δ-subsets (the countable intersections of open subsets) of $\mathbb{R}^n$ we use the representations θ^n and δ^n_G defined by $\theta(p) = U$, iff p is a list $J_0, J_1, \ldots$ of open balls from RB^n (encoded by I^n) such that $U = \bigcup_i J_i$ and $\delta^n_G\langle p_0,p_1,\ldots\rangle = \bigcap_j \theta^n(p_j)$ [23,22]. The θ^n-computable sets are called r.e.-open.

For the space $\mathrm{CP}(\mathbb{R}^m,\mathbb{R}^n)$ of the partial (topologically) continuous functions $f :\subseteq \mathbb{R}^m \to \mathbb{R}^n$, we use the multi-representation $\delta_{m,n}$ defined as follows: $f \in \delta_{m,n}(p)$ iff p is (encodes) a list $(J_i,K_i)_{i\in\mathbb{N}}$, ($J_i \in \mathrm{RB}^m$, $K_i \in \mathrm{RB}^n$), such that

$$f^{-1}L = \mathrm{dom}(f) \cap \bigcup\{J_i \mid K_i = L\} \qquad \text{for all } L \in \mathrm{RB}^n\,. \tag{3}$$

This representation is equivalent to $[\rho^m \to_p \rho^n]$ [10]. Therefore by (2), evaluation

$(f, x) \mapsto f(x)$ is $(\delta_{m,n}, \rho^m, \rho^n)$-computable.

If the representations of the sets under consideration are fixed, we will simply say "computable" instead of "(γ, δ)-computable" etc.

3 The Solution Operator Is Computable

By the Picard-Lindelöf theorem unique local solutions of the initial value problem (1) exist. The following version is from [12] slightly adjusted for our purposes. For $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ and $Z \subseteq \mathrm{dom}(f)$ we will call $M \in \mathbb{R}$ an upper bound of f on Z if $\|f(z)\| \leq M$ for all $z \in Z$, and we will call $L \in \mathbb{R}$ a Lipschitz constant of f on Z, if $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ for all $(t,x),(t,y) \in Z$.

Theorem 3.1 *[Picard-Lindelöf] Let $f : \overline{B}((t_0, x_0), r) \to \mathbb{R}^l$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^l$, $0 < r \leq 1$, be continuous. Let $L > 0$ be a Lipschitz constant and let $M \geq 1$ be an upper bound of f (on $\mathrm{dom}(f)$). Then the initial value problem (1) has a unique solution h on $[t_0 - b; t_0 + b]$ for $b = \min(r/M, 1/(2L))$ (see Figure 2).*

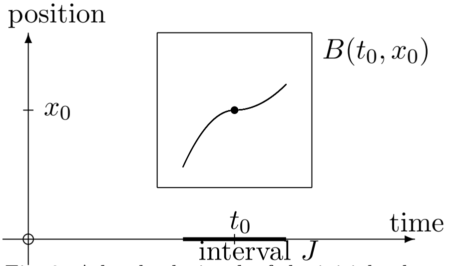

Fig. 2. A local solution h of the initial value problem (1).

We outline a classical proof [12], which already shows a way how to "compute" the local solution. Let $C(J)$ be the Banach space of continuous functions $f : J \to \mathbb{R}^l$, $J := [t_0 - b; t_0 + b]$, with maximum norm $\| \|_\infty$. Then $C_0 := \{g \in C(J) \mid \|g(t) - x_0\| \leq r$ for all $t \in J\}$ is a closed subset of $C(J)$ and the operator $A : C(J) \to C(J)$, defined by

$$A(g)(t) := x_0 + \int_{t_0}^{t} f(\tau, g(\tau))\, d\tau \,, \tag{4}$$

maps C_0 into itself and is contracting on C_0, that is, $\|A(g_1) - A(g_2)\|_\infty \leq \frac{1}{2}\|g_1 - g_2\|_\infty$ for $g_1, g_2 \in C_0$. By the Banach fixed point theorem the operator A has a unique fixed point, and this function is the local solution $h : [t_0 - b; t_0 + b] \to \mathbb{R}^l$ of our initial value problem [12]. The sequence $h_0, h_1, \ldots \in C_0$ defined by $h_0(t) := x_0$, $h_{n+1} := A(h_n)$, converges to the fixed point h of the operator A. Since $\|h_1 - h_0\|_\infty \leq r \leq 1$, $\|h_{n+1} - h_n\|_\infty \leq 2^{-n}$, and therefore, $\|h_k - h_n\|_\infty \leq 2^{-n+1}$ for $k > n$ and

$$\|h - h_n\|_\infty = \|h - A^n(g_0)\|_\infty \leq 2^{-n+1} \,. \tag{5}$$

Effectivizing this idea we get a fully uniform computable version of the Picard-Lindelöf theorem. For convenience we consider only positive integer bounds L and M.

Lemma 3.2 *[Computable Picard-Lindelöf] There is a $(\delta_{l+1,l}, \rho, \rho^l, \delta_{1,l})$-computable operator $T : (f, t_0, x_0) \mapsto h$ mapping each continuous function $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$, each $t_0 \in \mathbb{R}$ and each $x_0 \in \mathbb{R}^l$ to some $h :\subseteq \mathbb{R} \to \mathbb{R}^l$ such that the restriction of h to the interval $[t_0 - b\,; t_0 + b]$ is a local solution of (1), if for some r, $0 < r \leq 1$, and some natural numbers $M, L \geq 1$,*

(i) $\overline{B}((t_0, x_0), r) \subseteq \mathrm{dom}(f)$,

(ii) *L is a Lipschitz constant and M is an upper bound of f on $\overline{B}((t_0, x_0), r)$,*

(iii) $b = \min(r/M, 1/(2L))$.

In the following we will compute the global solution of the initial value problem (1) for locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain. Let h_0 be the local solution computed in Lemma 3.2 with initial point (t_0, x_0). Since $t_1 := t_0 + b \in \mathrm{dom}(f)$ we can extend the partial solution h_0 by a partial solution h_1 obtained by Lemma 3.2 with initial point (t_1, x_1) for $x_1 := h_0(t_1)$. This process can be iterated. For each of the points $(t_i, x_i) \in \mathrm{dom}(f)$ we need a neighbourhood ball with Lipschitz constant L_i and upper bound M_i. We consider a representation $\overline{\delta}$ such that a name of a function f contains data for evaluation (a $\delta_{l+1,l}$-name) and information about its open domain and local Lipschitz data. Local upper bounds M can be computed from these data (Lemma 3.4).

Definition 3.3 *Define a representation $\overline{\delta}$ of the locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain as follows: $f \in \overline{\delta}\langle p, q\rangle$, iff $f \in \delta_{l+1,l}(p)$ and $q \in \Sigma^\omega$ is (a code of) a sequence $((B_0, L_0), (B_1, L_1), \ldots)$, such that $B_i \in \mathrm{RB}^{l+1}$, $L_i \in \mathbb{N}$, $\overline{B}_i \subseteq \mathrm{dom}(f)$ and L_i is a Lipschitz constant of f on $\overline{B}_i$ for all $i \in \mathbb{N}$, and $\mathrm{dom}(f) = \bigcup_{i \in \mathbb{N}} B_i$.*

Obviously, $\overline{\delta} \leq \delta_{l+1,l}$, hence evaluation $(f, z) \mapsto f(z)$ is $(\overline{\delta}, \rho^{l+1}, \rho^l)$-computable. For applying Lemma 3.2 to $(t_0, x_0) \in \mathrm{dom}(f)$ we want to find a radius r' and constants L, M from the input data such that $\overline{B}((t_0, x_0), r') \subseteq \mathrm{dom}(f)$ and L is a Lipschitz constant and M is an upper bound of f on this closed ball. Since the sequence $(B_i, L_i)_i$ is not suitable for this purpose, we introduce another representation $\tilde{\delta}$ that is equivalent to $\overline{\delta}$. Let $B(x, r)/4 := B(x, r/4)$. Suppose, L is a Lipschitz constant and M is an upper bound of f on $\overline{B} \subseteq \mathrm{dom}(f)$. If $(t_0, x_0) \in B/4$ then L is a Lipschitz constant and M is an upper bound of f also on $\overline{B}((t_0, x_0), r/4) \in \mathrm{dom}(f)$. Therefore, a name of the new representation should also contain an upper bound of f on $\overline{B}_i \subseteq \mathrm{dom}(f)$ for each i and should satisfy the stronger condition $\mathrm{dom}(f) = \bigcup_{i \in \mathbb{N}} B_i/4$.

Lemma 3.4 *Define a representation $\tilde{\delta}$ of the locally Lipschitz bounded continuous functions $f :\subseteq \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ with open domain as follows: $f \in \overline{\delta}\langle p, q\rangle$, iff $f \in \delta_{l+1,l}(p)$ and $q \in \Sigma^\omega$ is (a code of) a sequence $((C_0, K_0, M_0), (C_1, K_1, M_1), \ldots)$ of triples such that for all i, $C_i \in \mathrm{RB}^{l+1}$ has radius ≤ 1, $\overline{C}_i \subseteq \mathrm{dom}(f)$, $K_i, M_i \in \mathbb{N} \setminus \{0\}$, and K_i is a Lipschitz constant and M_i is an upper bound of f on $\overline{C}_i$, and such that $\mathrm{dom}(f) = \bigcup_i C_i/4$. Then $\overline{\delta} \equiv \tilde{\delta}$.*

In the following lemma, we consider the behaviour of the global solution for for $t \geq t_0$. The case $t \leq t_0$ can be analysed similarly. Let T be the operator from

Lemma 3.2 for computing local solutions.

Lemma 3.5 *Let $f \in \tilde{\delta}\langle p, q\rangle$ where q is a name of the list $(C_i, K_i, M_i)_{i\in\mathbb{N}}$. Let h be the global solution for the initial condition (t_0, x_0) and suppose $t_0 < \overline{t}$. Then $h(t)$ exists for $t_0 \le t \le \overline{t}$, if, and only if, there are $n \ge 0$ and triples (D_k, L_k, N_k) $(k \le n)$ such that*

$$(D_0, L_0, N_0), \dots, (D_n, L_n, N_n) \in \{(C_i, K_i, M_i) \mid i \in \mathbb{N}\}\,, \tag{6}$$

$$(\forall k \le n)\,(t_k, x_k) \in D_k/4 \quad \textit{and} \tag{7}$$

$$\overline{t} < t_{n+1} \tag{8}$$

where the (t_k, x_k), $1 \le k \le n+1$, are determined as follows:

$$B(a_k, r_k) = D_k, \qquad d_k = \min\left(\frac{r_k}{4N_k}, \frac{1}{2L_k}\right), \tag{9}$$

$$t_{k+1} = t_k + d_k, \qquad x_{k+1} = T(f, t_k, x_k)(t_{k+1}). \tag{10}$$

By applying the above result we are now able to prove the following main result of this section.

Theorem 3.6 (i) *The solution operator $S : (f, t_0, x_0) \mapsto h$ where $h :\subseteq \mathbb{R} \to \mathbb{R}^l$ is the maximal solution of the initial value problem (1) is $(\overline{\delta}, \rho, \rho^l, \delta_{1,l})$-computable.*

(ii) *The function $F : (f, t_0, x_0) \mapsto U$ where U is the domain of the maximal solution of the initial value problem (1) is $(\overline{\delta}, \rho, \rho^l, \theta^1)$-computable.*

For open subsets of the real line, the function $U \mapsto \sup U$ is $(\theta, \overline{\rho}_>)$-computable and the function $U \mapsto \inf U$ is $(\theta, \overline{\rho}_>)$-computable. Therefore, from f and the initial values t_0, x_0 we can compute α from above and β from below such that (α, β) is the maximal interval of existence. If the input data are computable, α is right-r.e. and β is left-r.e.

4 The Complexity of Blowups

We will study the blowup for the initial value problem (1) for locally Lipschitz continuous functions $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$ defined on all of $\mathbb{R}^{l+1}$. We consider only "positive blowup", that is, the behaviour of the solution for $t \ge t_0$. Let BU_f be the set of all initial conditions (t_0, x_0) such that for the maximal solution h of (1), $\sup\{t \mid h(t) \text{ exists}\} < \infty$ (the *blowup points*) and let $\mathrm{CBU}_f := \mathbb{R}^{l+1} \setminus \mathrm{BU}_f$ be the set of initial conditions for which there is no blowup. By the next theorem the set CBU_f is a G_δ-set which can be computed from f.

Theorem 4.1 *The function $B : f \mapsto \mathrm{CBU}_f$ for locally Lipschitz continuous (total) functions $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is $(\overline{\delta}, \delta_G^{l+1})$-computable.*

Proof: Let S and F be the functions from Theorem 3.6. Define a function H_1 by

$$H_1(f, i, t_0, x_0) := \begin{cases} 1 \text{ if } \; i < \sup F(f, t_0, x_0) \\ 0 \text{ otherwise.} \end{cases}$$

This function is $(\overline{\delta}, \nu_{\mathbb{N}}, \rho, \rho^l, \overline{\rho}_<)$-computable: First compute the open set $V := F(f, t_0, x_0)$ and then try to find $i \in \mathbb{N}$ in V. As long as i has not been found print (a $\nu_{\mathbb{Q}}$-code of) 0 on the output tape, as soon as i has been found continue writing 1s. By (2), the function H_2 defined by $H_2(f, i)(t_0, x_0) := H_1(f, i, t_0, x_0)$ is $(\overline{\delta}, \nu_{\mathbb{N}}, [\rho^{l+1} \to \overline{\rho}_<])$-computable. $H_2(f, i)$ is the characteristic function of the set

$$V_i := \{(t_0, x_0) \mid S(f, t_0, x_0)(t) \text{ exists for some } t > i\}.$$

Define the Sierpinski representation $\delta^n_{\text{Sierpinski}}$ of subsets of $\mathbb{R}^n$ by

$$\delta^n_{\text{Sierpinski}}(p) = W \text{ iff } [\rho^n \to \overline{\rho}_<](p) \text{ is the characteristic function of } W.$$

Then $H_3 : (f, i) \mapsto V_i$ is $(\overline{\delta}, \nu_{\mathbb{N}}, \delta^{l+1}_{\text{Sierpinski}})$-computable. Since $\theta^n \equiv \delta^n_{\text{Sierpinski}}$ [6], the function H_3 is $(\overline{\delta}, \nu_{\mathbb{N}}, \theta^{l+1})$-computable. In particular, all the sets V_i are open. By (2), $H_4 : f \mapsto (V_i)_i$ is $(\overline{\delta}, [\nu_{\mathbb{N}} \to \theta^{l+1}])$-computable and hence $(\overline{\delta}, (\theta^{l+1})^\omega)$-computable. Since $\delta^n_G(p) = \bigcap_i (\theta^n)^\omega(p)(i)$, the function $H_5 : f \mapsto \bigcap_i V_i$ is $(\overline{\delta}, \delta^{l+1}_G)$-computable. It remains to observe that $\text{CBU}_f = \bigcap_i V_i$. □

Therefore, for every locally Lipschitz continuous (total) function $f : \mathbb{R}^{l+1} \to \mathbb{R}^l$, the set CBU_f is a G_δ-set and its complement BU_f is (by definition) an F_σ-set. If, in addition, f is computable (more precisely, $\overline{\delta}$-computable), then the set CBU_f is a computable G_δ-set.

Theorem 4.1 shows that F_σ is an upper complexity bound for the blowup sets. Although not every F_σ-set is a blowup set, for example, if it is bounded; this upper F_σ-complexity bound is sharp for $l \geq 2$. This result is a corollary of the following theorem in which we show that there is indeed a kind of G_δ lower bound of CBU_f for $l \geq 2$, even for time independent systems. We prove a non-uniform version. For a time independent system we may choose $t_0 = 0$.

Theorem 4.2 *For every computable G_δ-set $X \subseteq \mathbb{R}^{l-1}$ ($l \geq 2$) there is an effectively locally Lipschitz computable function $f : \mathbb{R}^l \to \mathbb{R}^l$ such that the solution u of the initial-value problem*

$$u'(t) = f(u(t)), \quad u(0) = (x_0, 0) \tag{11}$$

has a finite blowup for increasing t if and only if $x_0 \notin X$.

Proof. First we consider $l = 2$. For $n \in \mathbb{N}$, let $\text{BI}_n := \{(a \cdot 2^{-n}, (a+2) \cdot 2^{-n}) \mid a \in \mathbb{Z}\}$ and let I be a canonical injective numbering of the set $\text{BI} := \bigcup_n \text{BI}_n$ of "normed binary intervals". For an open real interval $(a; b)$ let $3(a; b) := (a-(b-a); b+(b-a))$. Let $X \subseteq \mathbb{R}$ be a computable G_δ-set. Then there is a computable function $g_0 : \mathbb{N}^2 \to \mathbb{N}$ such that

$$X = \bigcap_i \bigcup_j I g_0(i, j) .$$

As a first step we normalize this representation of X by an intersection of unions of open intervals.

Lemma 4.3 *There is a computable function $g : \mathbb{N}^2 \to \mathbb{N}$ such that for all $i, j \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$X = \bigcap_i O_i, \quad O_0 \supseteq O_1 \supseteq O_2 \dots \quad \textit{for} \quad O_i = \bigcup_j Ig(i,j)\,, \tag{12}$$

$$3Ig(i,j) \subseteq O_i\,, \tag{13}$$

$$\{j \mid x \in 3Ig(i,j)\} \quad \textit{is finite if} \quad x \in O_i\,. \tag{14}$$

Proof. (Lemma 4.3) Let $O_i := \bigcap_{i' \leq i} \bigcup_j Ig_0(i',j)$. Then

$$O_0 \supseteq O_1 \supseteq O_2 \dots\,, \quad X = \bigcap_i O_i \quad \text{and} \quad O_i = \bigcup_j Ig_1(i,j)$$

for some computable function g_1. Since every interval $K \in \mathrm{BI}$ is the union of intervals $L \in \mathrm{BI}$ such that $3L \subseteq K$, there is a computable function g_2 such that $O_i = \bigcup_j Ig_2(i,j)$ and $3Ig(i,j) \subseteq O_i$ for all i,j. Finally, by successively deleting for each i all $g_2(i,j)$ such that $Ig_2(i,j) \subseteq Ig_2(i,j')$ for some $j' < j$ (such intervals $Ig_2(i,j)$ are not necessary for generating O_i) we obtain a computable function g such that (12) and (13) and additionally

$$Ig(i,j) \not\subseteq Ig(i,j') \quad \text{if} \quad j' < j\,. \tag{15}$$

For showing (14) consider $x \in O_i$. Hence $x \in Ig(i,j_0) =: (c;d)$ for some j_0 and some c,d. Furthermore, $2^{-n} < \min(x-c, d-x)$ for some number n. Suppose, $x \in 3Ig(i,j)$ for infinitely many j. Since for each k there are at most 6 intervals $L \in \mathrm{BI}_k$ such that $x \in 3L$, there must be numbers $j > j_0$ and $m \geq n+3$ such that $x \in 3Ig(i,j)$ and $Ig(i,j) \in \mathrm{BI}_m$. Then $\mathrm{length}(3Ig(i,j)) = 6 \cdot 2^{-m} < 2^{-n}$ and hence $3Ig(i,j) \subseteq (c;d) = Ig(i,j_0)$ (since $x \in 3Ig(i,j)$). But this is false by (15), since $j_0 < j$. □(Lemma 4.3)

As a next step we define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$. For $k \in \mathbb{N}$, let $y_k := 2^{2^{2k}}$. Then

$$2 \leq y_k < y_k + y_k^2 + 2 \leq y_{k+1}\,. \tag{16}$$

For $k = \langle i,j \rangle$ let

$$g_k(x,y) := \begin{cases} y^2 - y_k^2 & \text{if } (x,y) \in Ig(i,j) \times (y_k; y_k^2) \\ 0 & \text{if } (x,y) \notin 3Ig(i,j) \times (y_k - 1; y_k^2 + 1)\,, \end{cases}$$

and for all (x,y) between the two rectangles, $g_k(x,y)$ is defined by linear interpolation such that $(k,x,y) \mapsto g_k(x,y)$ is a computable function that is effectively locally Lipschitz. By (16) $g_k(x,y) \cdot g_{k'}(x,y) = 0$ for $k \neq k'$. Define the function f by $f(x,y) := (f_1(x,y), f_2(x,y))$ where

$$f_1(x,y) := 0\,,$$

$$f_2(x,y) := \begin{cases} 1 & \text{if } y < 1 \\ y^2 - \sum_{k \in \mathbb{N}} g_k(x,y) & \text{else}\,. \end{cases}$$

We study the solution $u : \mathbb{R} \to \mathbb{R}^2$ of the initial-value problem (11). Since $f_1(x,y) = 0$, all trajectories are in y-direction, hence for each initial value $(x,0)$ we have a one-dimensional problem. We observe a particle starting at $(x, 0)$ traveling to position (x, y) with the prescribed speed $f_2(x, y)$ in y-direction. Since $f_2(x, y) \geq 1$ for all x, y, the particle will reach every point (x, y) for $y > 0$.

Suppose $x \in X$. By (12), for all i there is some j such that $x \in Ig(i, j)$. Therefore there are infinitely many k $(= \langle i, j\rangle)$ such that

$$f_2(x, y) = y^2 - g_k(x, y) = y_k^2 \;\text{ for }\; y_k \leq y \leq y_k^2.$$

Therefore, if $u(t_k) = (x, y_k)$ then $u(t_k + 1) = (x, y_k + y_k^2)$. Hence the particle needs one time unit for traveling from y_k to $y_k + y_k^2$. Since there are infinitely many such intervals, the particle cannot approach infinity in finite time. Therefore, there is no blowup for this initial value $(x, 0)$.

Suppose $x \notin X$. By (12) $x \in O_i$ only for finitely many i. By (14) for each of these numbers i, $x \in 3Ig(i, j)$ only for finitely many numbers j. Therefore, there are only finitely many $k = \langle i, j\rangle$ such that $g_k(x, y) > 0$ for some y. Hence, for some k, $f_2(x, y) = y^2$ for $y > y_k$. As is well known, in this case the particle will approach infinity in finite time. Therefore, there is a blowup for this initial value $(x, 0)$.

For $l > 2$, replace BI_n by $\mathrm{BI}_n^{(l)} := \{J_1 \times \ldots \times J_{l-1} \mid J_1, \ldots, J_{l-1} \in \mathrm{BI}_n\}$, replace the numbering I by a canonical numbering $I^{(l)}$ of $\mathrm{BI}^{(l)} := \bigcup_n \mathrm{BI}_n^{(l)}$, and define $3K$ accordingly for $K \in \mathbb{N}$. The rest of the proof remains unchanged. □

The above proof can be effectivized for showing that there is a $(\delta_G^{l-1}, \overline{\delta})$-computable multi-function mapping each G_δ-set to some locally Lipschitz function $f : \mathbb{R}^l \to \mathbb{R}^l$ such that the solution u of the initial-value problem "$u'(t) = f(u(t)),\;\; u(0) = (x_0, 0)$" has a finite blowup for increasing t, if and only if $x_0 \notin X$.

In the one-dimensional case, we can say even more if we restrict ourselves to functions f which do not depend on t. In this case the blowup sets do solely depend on the zeroes of f.

Theorem 4.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function. Then the "positive" blowup set BU_f of the initial value problem "$x'(t) = f(x(t)),\;\; x(0) = x_0$" is the union of two intervals $(-\infty, a)$ and (b, ∞) for some $a, b \in \overline{\mathbb{R}}$. If the function f is computable, then the constant a can be chosen to be $\overline{\rho}_<$-computable and the constant b to be $\overline{\rho}_>$-computable.*

Proof: If f has no zero, then there is a blowup either for all x_0 or for no x_0. In the first case let $a := -\infty$ and $b := \infty$, in the second case let $a := b :=:= \infty$.

Suppose that f has a zero. We observe that $x_0 \notin \mathrm{BU}_f$ if there are x_1, x_2 such that $f(x_1) = f(x_2) = 0$ and $x_1 \leq x_0 \leq x_2$.

If f has no greatest zero then let $b := \infty$. Suppose, f has a greatest zero β. Then there is a blowup either for all $x_0 > \beta$ or for no $x_0 > \beta$. In the first case let $b := \beta$, in the second case let $b := \infty$.

Correspondingly, if f has no smallest zero then let $a := -\infty$. Suppose, f has a smallest zero α. Then there is a blowup either for all $x_0 < \alpha$ or for no $x_0 < \alpha$. In the first case let $a := \alpha$, in the second case let $a := -\infty$.

By [23, Theorem 6.3.4] the smallest zero of a computable function (if it exists) is $\overline{\rho}_<$-computable and the greatest zero of a computable function (if it exists) is $\overline{\rho}_>$-computable. Furthermore, $-\infty$ and ∞ are $\overline{\rho}_<$-computable and $\overline{\rho}_>$-computable. $\square$

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